1. Second-Order Circuit: RLC Circuit Analysis

1. Be able to find Initial and Final Values for RLC Circuits.
   a. $v(0^-) = v(0^+) = v(0)$
   b. $i(0^-) = i(0^+) = i(0)$
   c. For a Series RLC Circuit with 1 Resistor, 1 Inductor, and 1 Capacitor.
      i. $L\frac{di(0)}{dt} + Ri(0) + v(0) = 0$
      ii. $\frac{di(0)}{dt} = \frac{1}{L}[-Ri(0) - v(0)] = -\frac{R}{L}\left[i(0) + \frac{v(0)}{R}\right] = -\frac{1}{\tau_L}\left[i(0) + \frac{v(0)}{R}\right]$
   d. For a Parallel RLC Circuit with 1 Resistor, 1 Inductor, and 1 Capacitor.
      i. $C\frac{dv(0)}{dt} + \frac{v(0)}{R} + i(0) = 0$
      ii. $\frac{dv(0)}{dt} = \frac{1}{C}\left[-\frac{v(0)}{R} - i(0)\right] = -\frac{1}{RC}[v(0) + Ri(0)] = -\frac{1}{\tau_C}[v(0) + Ri(0)]$
2. Be able to derive α and ω_0 for a Source Free Series RLC Circuit.
   a. Resistor, Inductor, and Capacitor have the -same Current.
   b. Use KVL to write a Differential Equation for the Circuit in terms of Current.
      i. $L\frac{d}{dt}i(t) + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau = 0$
   c. Take the Time Derivative to create a Second-order Differential Equation.
      i. $L\frac{d^2}{dt}i(t) + R\frac{d}{dt}i(t) + \frac{1}{C}i(t) = 0$
   d. Simplify so that the coefficient on the highest order term is "1".
      i. Divide the entire equation by L.
      ii. $\frac{d^2}{dt^2}i(t) + \frac{R}{L}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = 0$
   e. Write out the Characteristic Equation for our Differential Equation.
      i. $x(t) \to s^0 = 1$
      ii. $x'(t) = \frac{d}{dt}x(t) \to s^1$
      iii. $x''(t) = \frac{d^2}{dt^2}x(t) \to s^2$
      iv. $x^n(t) = \frac{d^n}{dt^n}x(t) \to s^n$ where $n \epsilon \mathbb{Z}^+$.
      v. Using the relationship between an arbitrary function $x(t)$ and s, we can write out the Characteristic Equation for our Differential Equation.
         1. $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
   f. Use the Quadratic Formula to solve for the Roots of the Characteristic Equation.
      i. If $f(x) = ax^2 + bx + c$ and we want the values for x which $f(x) = 0$…
         1. Then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
      ii. Applying the Quadratic Formula.
         1. $s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$
      iii. After simplifying the Quadratic Formula, we arrive at this conclusion.
         1. $s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$
   g. The Roots of our Characteristic Equation can be simplified further. This will be relating our values in the Quadratic Formula with a variable α and ω_0.

i. Let $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

ii. Using the following substitution, we arrive at the Roots of our Characteristic Equation in terms of α and ω_0.

1. $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

3. Be able to derive α and ω_0 for a Source Free Parallel RLC Circuit.
   a. Resistor, Inductor, and Capacitor have the same Voltage.
   b. Use KCL to write a Differential Equation for the Circuit in terms of Voltage.
      i. $C\frac{d}{dt}v(t) + \frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)\, d\tau = 0$
   c. Take the Time Derivative to create a Second-order Differential Equation.
      i. $C\frac{d^2}{dt^2}v(t) + \frac{1}{R}\frac{d}{dt}v(t) + \frac{1}{L}v(t) = 0$
   d. Simplify so that the coefficient on the highest order term is "1".
      i. Divide the entire equation by C.
      ii. $\frac{d^2}{dt^2}v(t) + \frac{1}{RC}\frac{d}{dt}v(t) + \frac{1}{LC}v(t) = 0$
   e. Write out the Characteristic Equation for our Differential Equation.
      i. Refer to (2.e.i-iv) to understand how to write out the Characteristic Equation for our Differential Equation.
         1. $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
   f. Use the Quadratic Formula to solve for the Roots of the Characteristic Equation.
      i. If $f(x) = ax^2 + bx + c$ and we want the values for x which $f(x) = 0$…
         1. Then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
      ii. Applying the Quadratic Formula.
         1. $s_{1,2} = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2}$
      iii. After simplifying the Quadratic Formula, we arrive at this conclusion.
         1. $s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$
   g. The Roots of our Characteristic Equation can be simplified further. This will be relating our values in the Quadratic Formula with a variable α and ω_0.
      i. Let $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$
      ii. Using the following substitution, we arrive at the Roots of our Characteristic Equation in terms of α and ω_0.
         1. $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
4. Be able to understand the characteristics of α and ω_0 for a Second-order Circuit.
   a. The general roots for a Second-order Circuit is defined as follows.
      i. $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
   b. There is a relationship between α and ω_0 for a Second-order Circuit. Depending on the number of Roots (2, 1, or 0) in our Characteristic Equation, this affects the Voltage and Current equation for our Second-order Circuit. There are 3 responses a RLC Circuit.
   c. Overdamped Response ($\alpha > \omega_0$)

i. This means that our Characteristic Equation has 2 Real Roots.
   1. Think of a quadratic function that intersects the x-axis 2 times.

ii. The Solution to our Second-order Differential Equation is the following.
   1. $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

iii. $x(t)$ will be our Time Varying Current for a Series RLC Circuit.

iv. $x(t)$ will be our Time Varying Voltage for a Parallel RLC Circuit.

v. Even if we have the solution, that is not enough to solve for our desired function. We have A_1 and A_2 in our solution so we will need to solve for these unknown coefficients.

vi. To solve for the coefficients, we will need a second equation to solve. In this case, we will take time derivative of our function $x(t)$.
   1. $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
      $x'(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$

vii. Now that we have 2 equations and 2 unknowns, we will use the initial conditions for our given RLC Circuit. We are working with a Second-order Circuit, so we need $x(0)$ and $x'(0)$ to solve for the coefficients. Refer to (1.a-d) to find the initial values for our given RLC Circuit.
   1. $x(0) = A_1 e^{s_1(0)} + A_2 e^{s_2(0)}$
      $x'(0) = s_1 A_1 e^{s_1(0)} + s_2 A_2 e^{s_2(0)}$

viii. We can simplify the previous Equation and we can solve the following System of Equations for the coefficients A_1 and A_2.
   1. $x(0) = A_1 + A_2$
      $x'(0) = s_1 A_1 + s_2 A_2$

d. Critically Damped Response ($\alpha = \omega_0$)

i. This means that our Characteristic Equation has 1 Real Root.
   1. Think of a quadratic function that intersects the x-axis 1 time.

ii. The Solution to our Second-order Differential Equation is the following.
   1. $x(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$

iii. $x(t)$ will be our Time Varying Current for a Series RLC Circuit.

iv. $x(t)$ will be our Time Varying Voltage for a Parallel RLC Circuit.

v. Even if we have the solution, that is not enough to solve for our desired function. We have A_1 and A_2 in our solution so we will need to solve for these unknown coefficients.

vi. To solve for the coefficients, we will need a second equation to solve. In this case, we will take time derivative of our function $x(t)$.

vii. This will be a little more difficult because we have 2 functions being multiplied with each other. We will use the Product Rule.
   1. If $x(t) = f(t)g(t)$, then $x'(t) = f(t)g'(t) + f'(t)g(t)$

viii. Using the Product Rule, we can take the time derivative of $x(t)$.
   1. $x(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$
      $x'(t) = -\alpha A_1 e^{-\alpha t} + A_2 e^{-\alpha t} - \alpha A_2 t e^{-\alpha t}$

ix. Now that we have 2 equations and 2 unknowns, we will use the initial conditions for our given RLC Circuit. We are working with a Second-order Circuit, so we need $x(0)$ and $x'(0)$ to solve for the coefficients. Refer to (1.a-d) to find the initial values for our given RLC Circuit.

1. $x(0) = A_1 e^{-\alpha(0)} + A_2 t e^{-\alpha(0)}$
   $x'(0) = -\alpha A_1 e^{-\alpha(0)} + A_2 e^{-\alpha(0)} - \alpha A_2 (0) e^{-\alpha(0)}$

x. We can simplify the previous Equation and we can solve the following System of Equations for the coefficients A_1 and A_2.
   1. $x(0) = A_1 + A_2$
      $x'(0) = -\alpha A_1 + A_2$

e. Underdamped Response ($\alpha < \omega_0$)
   i. This means that our Characteristic Equation has 0 Real Roots.
      1. Think of a quadratic function that never intersects the x-axis.
   ii. The Solution to our Second-order Differential Equation is the following.
      1. $x(t) = [A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]e^{-\alpha t}$
   iii. An Underdamped circuit has a Damped Frequency ω_d.
      1. $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
   iv. $x(t)$ will be our Time Varying Current for a Series RLC Circuit.
   v. $x(t)$ will be our Time Varying Voltage for a Parallel RLC Circuit.
   vi. Even if we have the solution, that is not enough to solve for our desired function. We have A_1 and A_2 in our solution so we will need to solve for these unknown coefficients.
   vii. To solve for the coefficients, we will need a second equation to solve. In this case, we will take time derivative of our function $x(t)$.
   viii. This will be a little more difficult because we have 2 functions being multiplied with each other. We will use the Product Rule.
      1. If $x(t) = f(t)g(t)$, then $x'(t) = f(t)g'(t) + f'(t)g(t)$
   ix. Using the Product Rule, we can take the time derivative of $x(t)$.
      1. $x(t) = [A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]e^{-\alpha t}$
         $x'(t) = -\alpha[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]e^{-\alpha t} +$
         $\omega_d[A_2 \cos(\omega_d t) - A_1 \sin(\omega_d t)]e^{-\alpha t}$
   x. Now that we have 2 equations and 2 unknowns, we will use the initial conditions for our given RLC Circuit. We are working with a Second-order Circuit, so we need $x(0)$ and $x'(0)$ to solve for the coefficients. Refer to (1.a-d) to find the initial values for our given RLC Circuit.
      1. $x(0) = \left[A_1 \cos\left(\omega_d(0)\right) + A_2 \sin\left(\omega_d(0)\right)\right]e^{-\alpha(0)}$
         $x'(0) = -\alpha\left[A_1 \cos\left(\omega_d(0)\right) + A_2 \sin\left(\omega_d(0)\right)\right]e^{-\alpha(0)} +$
         $\omega_d\left[A_2 \cos\left(\omega_d(0)\right) - A_1 \sin\left(\omega_d(0)\right)\right]e^{-\alpha(0)}$
   xi. We can simplify the previous Equation and we can solve the following System of Equations for the coefficients A_1 and A_2.
      1. $x(0) = A_1$
         $x'(0) = -\alpha A_1 + \omega_d A_2$