EE101 Lecture #25 May 12, 2018

- Quiz 9 today   QZ8 average = 8.37, $\sigma = 1.21$
- Course review on F, March 16
- Final Exam on T, March 20 4-7 p.m.

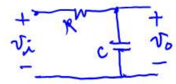
- Final Exam (12 problems)

  You are allowed to use 2 pages of formulas and tables only, not concepts, prob. soln

  Coverage: chapters 1-8, 13 (transformer, coupled inductors)
  14 (14.2 ~ 14.4 Bode plots)

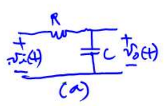
Chapter 14 Frequency Response

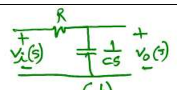
Transfer function of a RC circuit



$$\frac{1}{j} = \frac{1\times j}{j\times j} = \frac{j}{-1} = -j$$

	Time	Freq.	S-domain
R	$v = Ri$	$V(j\omega) = R\,I(j\omega)$	$V(s) = R\,I(s)$
L	$v = L\frac{di}{dt}$	$V(j\omega) = j\omega L\,I(j\omega)$ ELI	$V(s) = sL\,I(s)$
C	$v = \frac{1}{C}\int i(\tau)d\tau$	$V(j\omega) = \frac{1}{j\omega C}I(j\omega)$ ICE	$V(s) = \frac{1}{sC}I(s)$





From (a)

$$v_i(t) = R\left[C\frac{dv_o}{dt}\right] + v_o \xrightarrow{\text{Laplace transfor}} V_i(s) = RCs\,V_o(s) + V_o(s)$$

$$\underline{\frac{V_o(s)}{V_i(s)} = \frac{1}{1+sRC}}$$

For (b)

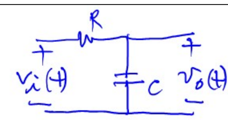
$$V_i(s) = \left[R + \frac{1}{cs}\right] I(s) \Rightarrow I(s) = \frac{V_i(s)}{R+\frac{1}{cs}}$$

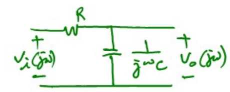
$$V_o(s) = I(s)\frac{1}{cs} = \frac{V_i(s)}{1+sRC}$$

$$\underline{\frac{V_o(s)}{V_i(s)} = \frac{1}{1+sRC}}$$

For this RC circuit

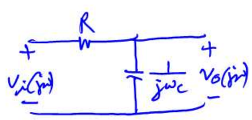
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1+sRC} \quad \text{(transfer function)}$$





$$V_o(j\omega) = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} V_i(j\omega) = \frac{1}{1+j\omega RC} V_i(j\omega)$$

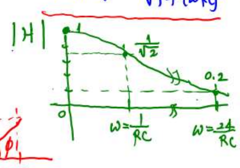
$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1+j\omega RC}$$ transfer function in Freq. domain

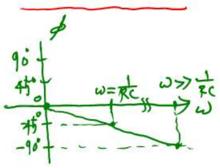


$$\frac{\alpha - j\beta}{(\alpha+j\beta)(\alpha-j\beta)} = \frac{\alpha-j\beta}{\alpha^2+\beta^2} = \frac{\alpha}{\alpha^2+\beta^2} - j\frac{\beta}{\alpha^2+\beta^2}$$

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1+j\omega RC} = \alpha + j\beta = |H(j\omega)|e^{j\phi(j\omega)} = |H(j\omega)|\angle\phi(j\omega)$$

$$\underline{|H(j\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}}}, \quad \underline{\phi(j\omega) = -\tan^{-1}\omega RC}$$





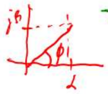


TABLE 14.1

For Example 14.1.

ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	−84°
1	0.71	−45°	20	0.05	−87°
2	0.45	−63°	100	0.01	−89°
3	0.32	−72°	∞	0	−90°

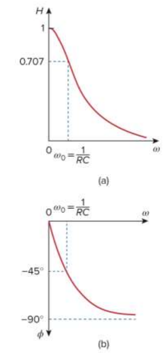
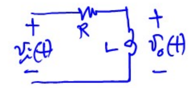
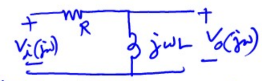


Figure 14.3 Frequency response of the RC circuit: (a) amplitude response, (b) phase response.

In Frequency domain



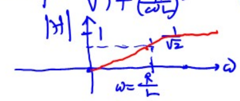


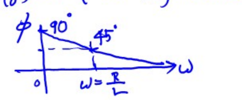
$$V_o(j\omega) = \frac{j\omega L}{R + j\omega L} V_i(j\omega)$$

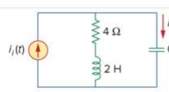
$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{j\omega L}{R + j\omega L} = \frac{1}{1 + \frac{R}{j\omega L}}$$

$$= \frac{1}{1 - j\frac{R}{\omega L}}$$

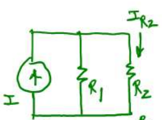
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{R}{\omega L}\right)^2}} \qquad \phi(j\omega) = 0 - \left(-\tan^{-1}\frac{R}{\omega L}\right) = +\tan^{-1}\frac{R}{\omega L}$$







$\frac{1}{j\omega\, 0.5} = -j\frac{2}{\omega}$



$I_{R_2} = I \frac{R_1}{R_1 + R_2}$

Figure 14.6 For **Example 14.2**.

Solution:

By current division,

current division law

$$\frac{4 + j2\omega}{4 + j2\omega - j\frac{2}{\omega}} = \quad \mathbf{I}_o(\omega) = \frac{\overset{R_1}{(4 + j2\omega)}}{\underbrace{(4 + j2\omega)}_{R_1} + \underbrace{(1/j0.5\omega)}_{R_2}} \mathbf{I}_i(\omega)$$

or

$$= \frac{4\omega + j2\omega^2}{4\omega + j2\omega^2 - j2} \qquad \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s + 2)}{s^2 + 2s + 1}, \qquad s = j\omega$$

$$H(j\omega) = \frac{-\omega^2 + j2\omega}{(1 - \omega^2) + j2\omega} \quad |H(j\omega)| = \frac{\sqrt{(-\omega^2)^2 + (2\omega)^2}}{\sqrt{(1-\omega^2)^2 + (2\omega)^2}} \quad \phi(j\omega) = \tan^{-1}\left(\frac{2\omega}{-\omega^2}\right) - \tan^{-1}\left(\frac{2\omega}{1 - \omega^2}\right)$$

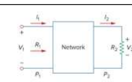


Figure 14.8 Voltage-current relationships for a four-terminal network.

$$G_{dB} = 10 \log_{10} \frac{P_2}{P_1} = 10 \log_{10} \frac{V_2^2/R_2}{V_1^2/R_1} \tag{14.8}$$

$$= 10 \log_{10} \left(\frac{V_2}{V_1}\right)^2 + 10 \log_{10} \frac{R_1}{R_2}$$

$$G_{dB} = 20 \log_{10} \frac{V_2}{V_1} - 10 \log_{10} \frac{R_2}{R_1} \tag{14.9}$$

Page 617

For the case when $R_2 = R_1$, a condition that is often assumed when comparing voltage levels, **Eq. (14.9)** becomes

$$\boxed{G_{dB} = 20 \log_{10} \frac{V_2}{V_1}} \tag{14.10}$$

Instead, if $P_1 = I_1^2 R_1$ and $P_2 = I_2^2 R_2$, for $R_1 = R_2$, we obtain

$$G_{dB} = 20 \log_{10} \frac{I_2}{I_1} \tag{14.11}$$

Three things are important to note from **Eqs. (14.5)**, **(14.10)**, and **(14.11)**:

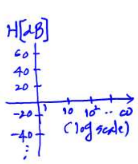
1. That 10 $\log_{10}$ is used for power, while 20 $\log_{10}$ is used for voltage or current, because of the square relationship between them ($P = V^2/R = I^2R$).
2. That the dB value is a logarithmic measurement of the *ratio* of one variable to another *of the same type*. Therefore, it applies in expressing the transfer function H in **Eqs. (14.2a)** and **(14.2b)**, which are dimensionless quantities, but not in expressing H in **Eqs. (14.2c)** and **(14.2d)**.
3. It is important to note that we only use voltage and current magnitudes in **Eqs. (14.10)** and **(14.11)**. Negative signs and angles will be handled independently as we will see in **Section 14.4**.

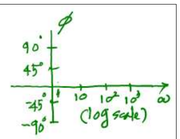
Bode plots

Bode plots are semilog plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency.

Page 618

Bode plots contain the same information as the nonlogarithmic plots discussed in the previous section, but they are much easier to construct, as we shall see shortly.





The transfer function can be written as

$$\mathbf{H} = H\underline{/\phi} = He^{j\phi} \tag{14.12}$$

Taking the natural logarithm of both sides,

$$\ln \mathbf{H} = \ln H + \ln e^{j\phi} = \ln H + j\phi \tag{14.13}$$

Thus, the real part of ln **H** is a function of the magnitude while the imaginary part is the phase. In a Bode magnitude plot, the gain

$$\boxed{H_{dB} = 20 \log_{10} H} \tag{14.14}$$

magnitude of $H(j\omega)$, i.e. $|H(j\omega)|$

is plotted in decibels (dB) versus frequency. **Table 14.2** provides a few values of H with the corresponding values in decibels. In a Bode phase plot, ϕ is plotted in degrees versus frequency. Both magnitude and phase plots are made on semilog graph paper.

TABLE 14.2

Specific gain and their decibel values.*

Magnitude H	$20 \log_{10} H$ (dB)
0.001	−60
0.01	−40
0.1	−20
0.5	−6
$1/\sqrt{2}$	−3
1	0

$20\log_{10} 0.001 = 20\log_{10}(10^{-3}) = -60\,dB$

$20\log_{10}\frac{1}{\sqrt{2}} = 20\log_{10} 2^{-\frac{1}{2}} = -10\log_{10} 2 = -3\,dB$

$$H(\omega) = \frac{N(\omega)}{D(\omega)} \longleftrightarrow H(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

A transfer function in the form of **Eq. (14.3)** may be written in terms of factors that have real and imaginary parts. One such representation might be

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots} \tag{14.15}$$

which is obtained by dividing out the poles and zeros in $\mathbf{H}(\omega)$. The representation of $\mathbf{H}(\omega)$ as in **Eq. (14.15)** is called the *standard form*. $\mathbf{H}(\omega)$ may include up to seven types of different factors that can appear in various combinations in a transfer function. These are:

1. A gain K
2. A pole $(j\omega)^{-1}$ or zero $(j\omega)$ at the origin
3. A simple pole $1/(1 + j\omega/p_1)$ or zero $(1 + j\omega/z_1)$
4. A quadratic pole $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ or zero $[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]$

$$H(j\omega) = K \frac{\left(1 + \frac{j\omega}{z_1}\right)\left(1 + \frac{j\omega}{z_2}\right)\cdots\left(1 + \frac{j\omega}{z_m}\right)}{\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)\cdots\left(1 + \frac{j\omega}{p_n}\right)}$$

for distinct m zeros & n poles

For double zeros, double poles exponents are 2 instead of 1.

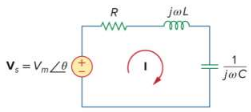


Figure 14.21 The series resonant circuit.

or

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{14.23}$$

Resonance results when the imaginary part of the transfer function is zero, or

$$\operatorname{Im}(\mathbf{Z}) = \omega L - \frac{1}{\omega C} = 0 \tag{14.24}$$

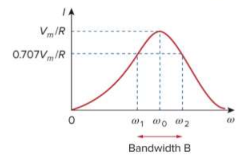
The value of ω that satisfies this condition is called the *resonant frequency* ω_0. Thus, the resonance condition is

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{14.28}$$

Page 629

is shown in **Fig. 14.22**; the plot only shows the symmetry illustrated in this graph when the frequency axis is a logarithm. The average power dissipated by the *RLC* circuit is

$$P(\omega) = \frac{1}{2} I^2 R \tag{14.29}$$



$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R} \tag{14.30}$$

At certain frequencies $\omega = \omega_1, \omega_2$, the dissipated power is half the maximum value; that is,

$$P(\omega_1) = P(\omega_2) = \frac{(V_m/\sqrt{2})^2}{2R} = \frac{V_m^2}{4R} \tag{14.31}$$

Hence, ω_1 and ω_2 are called the *half-power frequencies*.

The half-power frequencies are obtained by setting Z equal to $\sqrt{2}R$, and writing

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}\,R \tag{14.32}$$

Solving for ω, we obtain

$$\boxed{\begin{aligned}\omega_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\\ \omega_2 &= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\end{aligned}} \tag{14.33}$$

We can relate the half-power frequencies with the resonant frequency. From **Eqs. (14.26)** and **(14.33)**,

$$\omega_0 = \sqrt{\omega_1 \omega_2} \tag{14.34}$$

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} \tag{14.36}$$

Page 630

It is also regarded as a measure of the energy storage property of a circuit in relation to its energy dissipation property. In the series *RLC* circuit, the peak energy stored is $\frac{1}{2}LI^2$, while the energy dissipated in one period is $\frac{1}{2}(I^2R)(1/f_0)$. Hence,

$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f_0)} = \frac{2\pi f_0 L}{R} \tag{14.37}$$

or

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}} \tag{14.38}$$

Notice that the quality factor is dimensionless. The relationship between the bandwidth B and the quality factor Q is obtained by substituting **Eq. (14.33)** into **Eq. (14.35)** and utilizing **Eq. (14.38)**.

$$\boxed{B = \frac{R}{L} = \frac{\omega_0}{Q}} \tag{14.39}$$