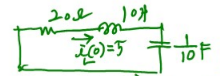
EE101 Lecture #23 May 7, 2018

- Problem solving session on March 17, Sat. The time and place to be announced
- Course review on March 16 (F)
- Final Exam (12 problems) $\frac{3\text{ hrs} = 180\text{ min}}{12} = \frac{15\text{ min}}{\text{prob.}}$

You are allowed to use 2 pages of formulas and tables only, not concepts, prob. soln

Coverage: chapters 1 - 8, 13 (transformer, coupled inductors)
14 (14.2 ~ 14.4 Bode plots)



$\frac{1}{10}$F $v_C(0) = 0$

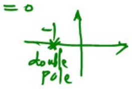
$$20 i_L + 10 \frac{di_L}{dt} + \frac{1}{(\frac{1}{10})}\int i_L(\tau)\, d\tau = 0$$

$$20 \frac{di_L}{dt} + 10 \frac{d^2 i_L(t)}{dt^2} + 10 i_L(t) = 0$$

For $i_L(t) = e^{st}$, characteristic eq. can be derived:

$$20 s e^{st} + 10 s^2 e^{st} + 10 e^{st} = 0$$

$$\Rightarrow s^2 + 2s + 1 = (s+1)^2 = 0$$

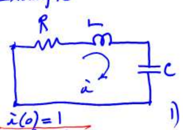


$$i_L(t) = A_1 t e^{-t} + A_2 e^{-t}$$

- $i_L(0) = A_2 = 5$
- $i_L'(0) = A_1 e^{-t} + A_1 t(-1) e^{-t} - A_2 e^{-t}\Big|_{t=0}$
  $= A_1 - A_2 = 0$ $\boxed{A_1 = A_2}$

$$i_L(t) = 5 t e^{-t} + 5 e^{-t} = 5(t+1) e^{-t}$$ ans

Example



$R = 40\,\Omega$
$L = 4$ H
$C = \frac{1}{4}$ F

$$\begin{cases} Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0 \\ R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i = 0 \\ (Ls^2 + Rs + \frac{1}{C}) I(s) = F(s) \\ I(s) = \frac{F(s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \end{cases}$$

$i(0) = 1$
$v(0) = 0$
$40 + 4\frac{di(0)}{dt} + 0 = 0$
$i'(0) = -10$

1) Calculate the characteristic roots of the circuit. $s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 10s + 1 = 0$

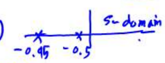
2) Is it over damped, underdamped or critically damped?

$$\boxed{\begin{aligned} &(s+5)^2 - 24 = 0 \\ &s = -5 \pm \sqrt{24} \\ &= -5 \pm 2\sqrt{6} \\ &= -5 \pm 2(2.45) \\ &= -0.5, -9.5 \end{aligned}}$$

$$s_1, s_2 = -\underbrace{\frac{R}{2L}}_{\alpha} \pm \sqrt{\left(\underbrace{\frac{R}{2L}}_{\alpha}\right)^2 - \underbrace{\frac{1}{LC}}_{\omega_0^2}} = -5 \pm \sqrt{5^2 - 1}$$

$$\left[\begin{aligned} \frac{R}{2L} &= \frac{40}{2\times 4} = 5 = \alpha \\ \frac{1}{LC} &= \frac{1}{4\times\frac{1}{4}} = 1 = \omega_0^2 \end{aligned}\right]$$

$\alpha^2 - \omega^2 = 25 - 1 = 24 > 0$ (overdamped)



$$i(t) = A_1 e^{-0.5t} + A_2 e^{-9.5t}$$

$$\begin{cases} i(0) = A_1 + A_2 \\ i'(0) = -0.5A_1 - 9.5A_2 = -10 \end{cases}$$

$A_1 + A_2 = 1$ (1)
$-0.5A_1 - 9.5A_2 = -10$ (2)
$\to -A_1 - 19A_2 = -20$ (2)'
(1)+(2)' $\to 0 - 18A_2 = -19$ $A_2 = \frac{19}{18}$
From (1), $A_1 = -\frac{1}{18}$

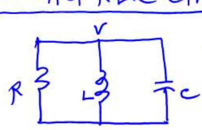
$$i(t) = -\frac{1}{18} e^{-0.5t} + \frac{19}{18} e^{-9.5t}$$ ans

$i(0) = -\frac{1}{18} + \frac{19}{18} = 1$ ✓

$i'(0) = +\frac{0.5}{18} - \frac{19 \times 9.5}{18} = \frac{180.5}{18}$ ... $= -\frac{180}{18}$
$= -10$ ✓

Both check out, thus the ans above is correct ☒

Parallel RLC Circuit



In series RLC circuit the loop current is common to all elements. (same)
In parallel RLC circuit, the node voltage is common. (same)

KCL $\Rightarrow \frac{v}{R} + \frac{1}{L}\int v(\tau)\,d\tau + C\frac{dv}{dt} = 0$

Taking $\frac{d}{dt} \to \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0$

dividing by C $\to \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$

For $v = e^{st}$, $\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right]e^{st} = 0$

Thus $\boxed{s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0} \leftrightarrow \boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$ series ckt

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$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

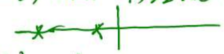
Let $\alpha = \frac{1}{2RC}$, $\omega_0 = \frac{1}{\sqrt{LC}}$

Then $s^2 + 2\alpha s + \omega_0^2 = 0$

$(s+\alpha)^2 - (\alpha^2 - \omega_0^2) = 0$

$(s + \alpha + \sqrt{\alpha^2 - \omega_0^2})(s + \alpha - \sqrt{\alpha^2 - \omega_0^2}) = 0$ $\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$

1) If $\alpha > \omega_0$, then s_1, s_2 are both negative, real.

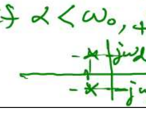


Overdamped

2) If $\alpha = \omega_0$, then $s_1, s_2 = -\alpha$ (double root)

critically damped

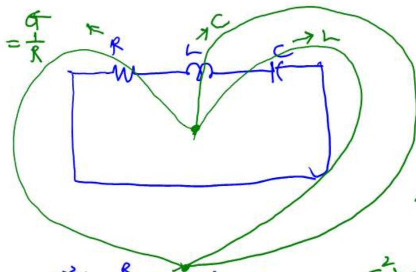
3) If $\alpha < \omega_0$, then $s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$
$= -\alpha \pm j\omega_d$

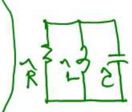


underdamped

Dual circuit

$G = \frac{1}{R}$





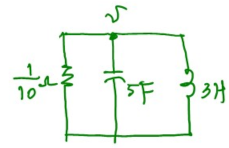
Dual of series RLC

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0 \qquad s^2+\frac{1}{\hat R \hat C}s+\frac{1}{\hat L \hat C}=0$$

$$\frac{R}{L}=\frac{1}{\hat R\hat C} \qquad \frac{1}{LC}=\frac{1}{\hat L\hat C}$$

$$\Rightarrow \boxed{\hat C = L,\ \hat R=\frac{1}{R},\ \hat L = C}$$

Example



$$10i+5\frac{di}{dt}+\frac{1}{3}\int i(\tau)d\tau=0$$

$$\boxed{5\frac{d^2i}{dt^2}+10\frac{di}{dt}+\frac{1}{3}i=0}$$

$$5s^2+10s+\frac{1}{3}=0$$

$$\frac{v}{(1/10)}+5\frac{dv}{dt}+\frac{1}{3}\int v(\tau)d\tau=0$$

$$10v+5\frac{dv}{dt}+\frac{1}{3}\int v(\tau)d\tau=0$$

$$\boxed{5\frac{d^2v}{dt^2}+10\frac{dv}{dt}+\frac{1}{3}v(t)=0}$$

$\longleftrightarrow$ same! $5s^2+10s+\frac{1}{3}=0$

TABLE 8.1

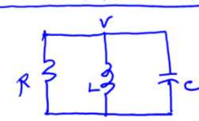
Dual pairs.

Resistance R	Conductance G
Inductance L	Capacitance C
Voltage v	Current i
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Two circuits that are described by equations of the same form, but in which the variables are interchanged, are said to be dual to each other.

Two circuits are said to be **duals** of one another if they are described by the same characterizing equations with dual quantities interchanged.

Parallel RLC Circuit (recalled)



In series RLC circuit the loop current is common to all elements. (same)

In parallel RLC circuit, the node voltage is common. (same)

$$KCL \Rightarrow \frac{v}{R}+\frac{1}{L}\int v(\tau)d\tau + C\frac{dv}{dt}=0$$

$$\text{Taking } \frac{d}{dt} \rightarrow \frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v+C\frac{d^2v}{dt^2}=0$$

$$\text{dividing by } C \rightarrow \frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

$$\text{For } v=e^{st},\quad \left[s^2+\frac{1}{RC}s+\frac{1}{LC}\right]e^{st}=0$$

$$\text{Thus } \boxed{s^2+\frac{1}{RC}s+\frac{1}{LC}=0} \leftrightarrow \underset{\text{series ckt}}{\boxed{s^2+\frac{R}{L}s+\frac{1}{LC}=0}}$$

Give the characteristic eq. of the parallel RLC circuit as $s^2+\frac{1}{RC}s+\frac{1}{LC}=0$

$\Updownarrow$

$\left(s^2+\frac{R}{L}s+\frac{1}{LC}=0 \text{ for the series RLC}\right)$

$$\left(s+\frac{1}{2RC}\right)^2-\left(\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}\right)=0$$

$$s_{1,2}=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

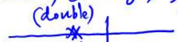
$$= -\alpha \pm\sqrt{\alpha^2-\omega_0^2}$$

Let $\frac{1}{LC}=\omega_0^2$, $\frac{1}{2RC}=\alpha$

case 1) $\alpha>\omega_0$, s_1, s_2 are neg. real

$$\Rightarrow v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

case 2) $\alpha=\omega_0$, $s_1=s_2=$ (double)

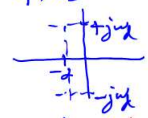


$$v(t)=A_1e^{-\alpha t}+A_2te^{-\alpha t}$$

case 3) $\alpha<\omega_0$

$$s_1, s_2 = -\alpha\pm j\sqrt{\omega_0^2-\alpha^2}=-\alpha\pm j\omega_d$$

$$\omega_d=\sqrt{\omega_0^2-\alpha^2}$$



$$v(t)=e^{-\alpha t}M\cos(\omega_dt-\theta)$$

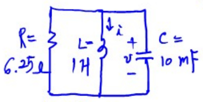
$$v(0)=M\cos(-\theta)=M\cos\theta \qquad (A)$$

$$v'(0)=-\alpha e^{-\alpha t}M\cos(\omega_dt-\theta)+e^{-\alpha t}M(-\omega_d\sin(\omega_dt-\theta))\Big|_{t=0}$$

$$=\underbrace{-\alpha M\cos\theta}_{=v(0)}+\omega_dM\sin\theta \qquad (B)$$

$$\frac{v_0'(0)}{v_0}=-\alpha+\omega_d\tan\theta \Rightarrow \theta, \text{ next use (A) to find } M$$

Example



$i(0) = 0$
$v(0) = 5$

$\omega_0^2 = \frac{1}{LC} = \frac{1}{1\times 10\times 10^{-3}} = 100, \ \underline{\omega_0 = 10}$

$\alpha = \frac{1}{2RC} = \frac{1}{2(6.25)10\times 10^{-3}} = \frac{1000}{125} = \underline{8}$

$\alpha < \omega_0$ (case 3)!

$$v(t) = e^{-\alpha t} M \cos(\omega_d t - \theta)$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{100 - 8^2} = 6$

$$v(t) = e^{-8t} M \cos(6t - \theta)$$

next identify M, θ from initial conditions

$v(0) = M\cos\theta = 5$

$C\frac{dv(t)}{dt}\Big|_{t=0} = -\left(\frac{v(t)}{6.25} + i(t)\right)\Big|_{t=0}$

$v'(0) = -\frac{1}{C}\left(\frac{v(0)}{6.25} + 0\right) = -\frac{1}{10\times 10^{-3}}\left(\frac{5}{6.25}\right) = -80$

$v'(t) = -\alpha e^{-\alpha t} M\cos(\omega_d t - \theta) + e^{-\alpha t} M(-\alpha)\sin(\omega_d t - \theta)$

$\Rightarrow v'(0) = -\alpha \underbrace{M\cos\theta}_{=v(0)=5} + \alpha M \sin\theta$

$\frac{v'(0)}{5} = -\alpha + \alpha\tan\theta$

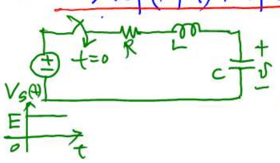
$\frac{-80}{5} = -8 + 8\tan\theta$

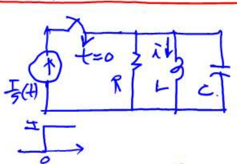
$-8 = 8\tan\theta \Rightarrow \tan\theta = -1 \Rightarrow \theta = -45^\circ$

thus, $M\cos\theta = 5, \ M = \frac{5}{\cos(-45^\circ)} = \frac{5}{\sqrt{2}}$

$$\underline{v(t) = \frac{5}{\sqrt{2}} e^{-6t}\cos(6t + 45^\circ)}$$

Step (input) Response of RLC Circuits.





For $t > 0$

$E = L\frac{d}{dt}\left(C\frac{dv}{dt}\right) + R\left(C\frac{dv}{dt}\right) + v$

$= LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v$

$$\boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{E}{LC}}$$

$v(t)$ w/o input

In DC steady state $v_{ss} = E$ & $v(t) = v_t + v_{ss}$

$I_s = \frac{L\frac{di}{dt}}{R} + i + C\frac{d}{dt}\left(L\frac{di}{dt}\right)$

$= LC\frac{d^2i}{dt^2} + \frac{L}{R}\frac{di}{dt} + i$

$$\boxed{\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + i = \frac{I_s}{LC}}$$

Series RLC

$v(t) = v_t + v_{ss} = E$

v_t natural response due to initial condition

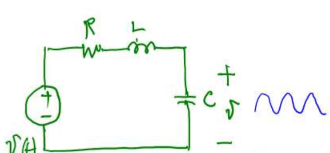
i) $\omega_0 > \alpha_0$
ii) $\omega_0 = \alpha_0$
iii) $\omega_0 < \alpha_0$

Parallel RLC

$i(t) = i_t(t) + i_{ss} = I_s$

$i_t(t)$ natural response due to initial condition

i) $\omega_0 > \alpha_0$
ii) $\omega_0 = \alpha_0$
iii) $\omega_0 < \alpha_0$



$v_s(t) = LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v$

$v = v_t + v_{ss}$

when $v_s(t) = 0$, $\underline{R = 0}$

$$\boxed{LC\frac{d^2v}{dt^2} + v = 0}$$

$\frac{d^2v}{dt^2} + \underbrace{\left(\frac{1}{LC}\right)}_{=\omega_0^2}v = 0$

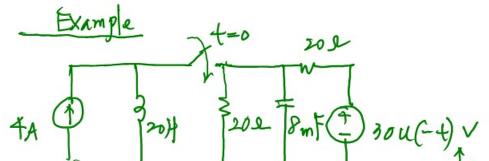
Pendulum

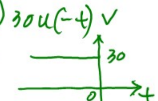
$$\boxed{\text{Pendulum eq} \quad Ml\frac{d^2\theta}{dt^2} + Mg\sin\theta = 0}$$

For small θ, $\sin\theta \approx \theta$

$\frac{1}{LC} \leftrightarrow \frac{g}{l} \qquad \frac{d^2\theta}{dt^2} + \left(\frac{g}{l}\right)\theta = 0$

Example

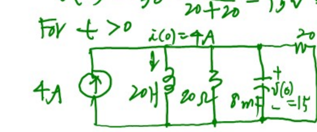




$i(0_-) = 4A = i(0_+)$

$v(0_-) = 30\frac{20}{20+20} = 15V = v(0_+)$

For $t > 0$



$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\times 8\times 10^{-3}}} = \frac{10}{4} = 2.5$

$\alpha = \frac{R}{2L} = \frac{10}{40} = 0.25$

$\omega_0 > \alpha$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 2.4875$

$$i_t(t) = \underline{Me^{-\alpha t}\cos(\omega_d t - \theta)}$$

$$i_t(t) = e^{-\alpha t} M \cos(\omega_d t - \theta) \qquad \omega_d = 2.4875$$

$$= 4.16\, e^{-0.5t} \cos(\omega_d t - 15.5^\circ)$$

$$i_t(0) = M\cos(-\theta) = M \underline{\cos\theta} = 4 \quad \left[\, M = \frac{4}{0.962} = \underline{4.16} \right.$$

$$i_t'(t)\Big|_{t=0} = -\alpha e^{-\alpha t} M \cos(\omega_d t - \theta) + e^{-\alpha t} M(-\omega_d \sin(\omega_d t - \theta))\Big|_{t=}$$

$$= -\alpha \underbrace{M\cos\theta}_{=4} + \omega_d M \sin\theta = i_t'(0)$$

$$L i_t'(0) = v(0) = 15 \qquad i_t'(0) = \frac{15}{20} = 0.75$$

$$\frac{i_t'(0)}{4} = -\alpha + \omega \tan\theta = -0.5 + 2.4875 \tan\theta$$

$$\tan\theta = \left(\frac{0.75}{4} + 0.5\right) / 2.4875 \simeq 0.276$$

$\theta \simeq 0.27\ \text{rad}$ $\qquad \cos\theta = 0.962$