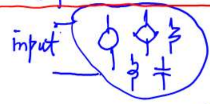
EE101 Lecture #22 May 5, 2018

Quiz 8 today

Quiz 7 { Average 6.5, σ 1.15

chap 8 Second-order circuits



circuit contains both capacitor and inductor elements.

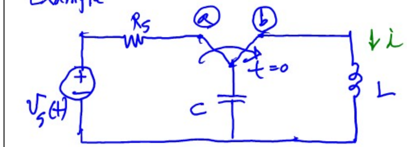
Circuit equations become 2nd-order differential eq.

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f(t)$$

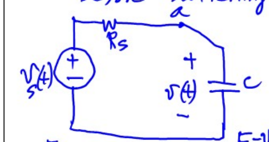
where x = either voltage across a capacitor or a current through an inductor

a, b, c depends on R, L, C & dep. sources

Example

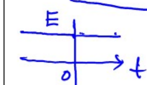


Before switching from (a) to (b), $i=0$



KCL at (a)

$$\frac{E-v}{R_S}=C\frac{dv}{dt}$$



$$E-v=R_S\left(C\frac{dv}{dt}\right)$$

$$E-v=R_SC\left(-\frac{d(E-v)}{dt}\right)$$

$$\frac{d(E-v)}{(E-v)}=-\frac{1}{R_SC}dt$$

$$\frac{d(E-v)}{(E-v)}=-\frac{1}{R_SC}dt$$

Integrating both sides, $\int\frac{dx}{x}=\ln x$

$$\ln(E-v)\Big|_{E-v(-\infty)}^{E-v(t)}=-\frac{1}{R_SC}(t-t_{-\infty})$$

and

$$e^{\ln\frac{E-v(t)}{E-v(-\infty)}}=e^{-\frac{1}{R_SC}(t-t_{-\infty})}$$

$$\Rightarrow E-v(t)=(E-v(-\infty))\,e^{-\frac{1}{R_SC}(t-t_{-\infty})}$$

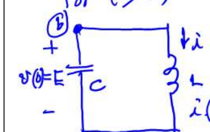
$$v(t)=E-(E-v(-\infty))\,e^{-\frac{1}{R_SC}\underbrace{(t-t_{-\infty})}_{=\infty \text{ at } t=0}}$$

At $t=0$ (just before switching from (a) to (b)),

$$v(0)=E-(E-v(-\infty))\,e^{-\frac{1}{R_SC}(\infty)}=E$$

For $t>0$, we have a parallel LC circuit (oscillator)

KCL at (b)



$$C\frac{dv}{dt}+i=0 \quad (1)$$

$$L\frac{di}{dt}=v \quad (2)$$

$$\frac{d}{dt}(1)\rightarrow C\frac{d^2v}{dt^2}+\frac{di}{dt}=0 \quad (3)$$

$$(2)\,\&\,(3)\rightarrow C\frac{d^2v}{dt^2}+\frac{v}{L}=0$$

$$\boxed{\frac{d^2v}{dt^2}+\frac{1}{LC}v=0}$$

Initial conditions

$v(0_-)=v(0_+)=E$

$L\frac{di}{dt}=v$, $\left.\frac{di}{dt}\right|_{t=0-}=\frac{E}{L}$

$\frac{di(0_-)}{dt}=\frac{di(0_+)}{dt}=\frac{E}{L}$

$i(0_-)=i(0_+)=0=-C\frac{dv(0_-)}{dt}=-C\frac{dv(0_+)}{dt}$

To derive a characterist equation of the circuit, put $v(t)=e^{st}$, then

$$\frac{d^2v}{dt^2}+\frac{1}{LC}v=0\rightarrow s^2e^{st}+\frac{1}{LC}e^{st}=0$$

$$\rightarrow\boxed{s^2+\frac{1}{LC}=0}\quad s^2=-\frac{1}{LC}$$

$$s_1,s_2=\pm\sqrt{\frac{-1}{LC}}=\pm j\frac{1}{\sqrt{LC}}$$

$j=\sqrt{-1}$

$\frac{1}{\sqrt{LC}}=\omega_0$

$$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

$$=A_1e^{+j\omega_0t}+A_2e^{-j\omega_0t}$$

$v(0)=A_1+A_2=E$

$v'(0)=0=+j\omega_0A_1e^{+j\omega_0(0)}-j\omega_0A_2e^{-j\omega_0(0)}$

$=+j\omega_0A_1-j\omega_0A_2=0 \Rightarrow A_1=A_2$

$\Rightarrow A_1=A_2=\frac{E}{2}$

$$\Rightarrow v(t)=\frac{E}{2}\left[e^{j\omega_0t}+e^{-j\omega_0t}\right]$$

$$=E\left(\frac{e^{j\omega_0t}+e^{-j\omega_0t}}{2}\right)=E\,\cancel{\sin}\cos\omega_0t$$

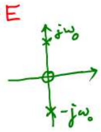
ans

Laplace Transform method (Ref Table 15.1 textbook)

$$s^2 V(s) - s v(0+) - v'(0+) + \frac{1}{LC} V(s) = 0$$

$$\left(s^2 + \frac{1}{LC}\right) V(s) = s v(0+) + v'(0+)$$
$$= sE + 0$$

$$V(s) = \frac{s}{s^2+\frac{1}{LC}} E = \frac{s}{s^2+\omega_0^2} E \qquad \left(\omega_0 = \frac{1}{\sqrt{LC}}\right)$$



$\downarrow \mathcal{L}^{-1}$

$$v(t) = E\cos\omega_0 t$$

$v(0) = E$ (✓)

$v'(0) = -\omega_0 E \sin\omega_0 t\big|_{t=0_+} = 0$ (✓)

Solution of $\frac{d^2v}{dt^2} + \frac{1}{LC} v = 0$

when $v(t) = A\cos(\omega t+\theta)$

$$\frac{dv}{dt} = -\omega A \sin(\omega t+\theta)$$
$$\frac{d^2v}{dt^2} = -\omega^2 A\cos(\omega t+\theta)$$

$\Rightarrow -\omega^2 A\cos(\omega t+\theta) + \frac{1}{LC}(A\cos(\omega t+\theta)) = 0$

$\Rightarrow \omega^2 = \frac{1}{LC} \Rightarrow \omega = \pm\frac{1}{\sqrt{LC}}$

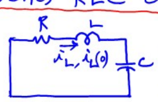
$\omega_0 = \frac{1}{\sqrt{LC}}$

$v(t) = A\cos\left(\frac{1}{\sqrt{LC}}t+\theta\right)$

$v(0) = E = A\cos\theta$ for any $\theta \Rightarrow \theta = 0, A = E$

$\rightarrow \boxed{v(t) = E\cos\omega_0 t}$

Series RLC circuit



$v_C(0) = \frac{1}{C}\int_{-\infty}^{0} i_C(\tau)\,d\tau$ (1)

$i_C(t) = i_L(t)$ (2)

$i_L(t)\big|_{t=0} = i_L(0)$

KVL, $R i_L + L\frac{di_L}{dt} + \frac{1}{C}\int_{-\infty}^{t} i_L(\tau)\,d\tau$ (3)

Taking $\frac{d}{dt}$ on both sides,

$L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{1}{C} i_L(t) = 0$ (4)

At $t=0$, from (3) $R i_L(0) + L\frac{di_L(0)}{dt} + v_C(0) = 0$

$\frac{di_L(0)}{dt} = -\frac{1}{L}\left(R i_L(0) + v_C(0)\right)$ (5)

For $i_L(t) = Ae^{st}$,

(4) $\rightarrow L s^2 A e^{st} + R s A e^{st} + \frac{1}{C} A e^{st} = 0$

$\Rightarrow (Ls^2 + Rs + \frac{1}{C}) = 0$ or $\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$

Characteristic Equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\left(\frac{R}{2L}\right)s + \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right] + \left(\frac{R}{2L}\right)^2 = 0$$

$$\left(s + \frac{R}{2L}\right)^2 - \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right] = 0$$

$$\left(s + \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)\left(s + \frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = 0$$

$$\Rightarrow s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Let $\frac{R}{2L} = \alpha$, $\frac{1}{LC} = \omega_0^2$, then

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$$

1) If $\alpha > \omega_0$ $\left(\frac{R}{2L} > \frac{1}{\sqrt{LC}}\right)$ then $s_{1,2}$ both are negative real (over damped)
2) If $\alpha = \omega_0$ $s_{1,2} = -\alpha$ (critically damped)
3) If $\alpha < \omega_0$ $s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$ (under damped)

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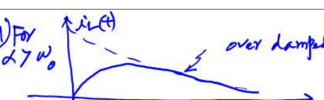
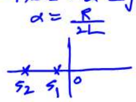
1) For $\alpha > \omega_0$



$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

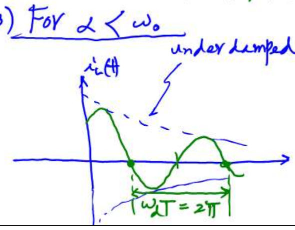
$\alpha = \frac{R}{2L}$



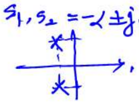
solution $i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$\left.\begin{array}{l} i_L(0) = A_1 + A_2 \\ i_L'(0) = A_1 s_1 + A_2 s_2 \end{array}\right\}$ solve for A_1, A_2

3) For $\alpha < \omega_0$



$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$



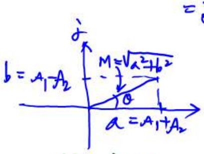
$i_L(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$

$\sqrt{\omega_0^2 - \alpha^2} = \omega_d$

$$= e^{-\alpha t}\left[A_1 e^{+j\omega_d t} + A_2 e^{-j\omega_d t}\right]$$

$$= e^{-\alpha t}\left[A_1\cos\omega_d t + jA_1\sin\omega_d t + A_2\cos\omega_d t - jA_2\sin\omega_d t\right]$$

$$= e^{-\alpha t}\left[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\right]$$



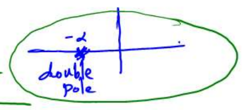
$$= e^{-\alpha t}\left[M\cos\theta\cos\omega_d t + jM\sin\theta\sin\omega_d t\right]$$

$$= e^{-\alpha t} M\cos(\omega_d t - \theta) = i_L(t)$$

Use $\left.\begin{array}{l} i_L(0) = A_1 + A_2 \\ i_L'(0) = A_1 s_1 + A_2 s_2 \end{array}\right\}$ solve for A_1, A_2

3) For $\alpha = \omega_0$

$s_1, s_2 = -\alpha$



$$i_L(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$

$i_L(0) = A_2$

$i_L'(0) = A_1 e^{-\alpha t} + A_1 t(-\alpha) e^{-\alpha t} - A_2 \alpha e^{-\alpha t} \Big|_{t=0}$

$= A_1 - A_2\alpha \Rightarrow i_L'(0) = A_1 - i_L(0)\alpha$

$\Rightarrow A_1 = i_L'(0) + i_L(0)\alpha$

An Alternative Approach (using Laplace Transform)

$L[s^2 I_L(s) - s\, i_L(0) - i_L'(0)] + R[s\, I_L(s) - i_L(0)] + \frac{1}{C} I_L(s) = 0$

where $i_L(0)=0$

$(Ls^2 + Rs + \frac{1}{C}) I_L(s) = L s\, i_L(0) + L i_L'(0) + R i_L(0)$

$$I_L(s) = \frac{Ls\, i_L(0) + L i_L'(0) + R i_L(0)}{Ls^2 + Rs + \frac{1}{C}} = \frac{s\, i_L(0) + i_L'(0) + \frac{R}{L} i_L(0)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

double pole $\frac{as + b}{(s+\alpha)^2} = \frac{K_1}{(s+\alpha)^2} + \frac{K_2}{(s+\alpha)}$ (*)

$a = i_L(0)$, $b = i_L'(0) + \frac{R}{L} i_L(0) = -v_C(0)/L$

$$I_L(s) = \frac{K_1}{(s+\alpha)^2} + \frac{K_2}{s+\alpha}$$

$\downarrow \mathcal{L}^{-1}$

$$i_L(t) = K_1 t e^{-\alpha t} + K_2 e^{-\alpha t} \quad \text{Eq}(**)$$

- $i_L(0) = K_2$
- $i_L'(0) = K_1 e^{-\alpha t} + K_1 t(-\alpha) e^{-\alpha t} - K_2 \alpha e^{-\alpha t} \Big|_{t=0}$

$= K_1 - K_2\alpha$

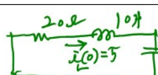
thus $K_1 = i_L'(0) + K_2\alpha = i_L'(0) + i_L(0)\alpha$

$$i_L(t) = [i_L'(0) + i_L(0)\alpha]\, t e^{-\alpha t} + i_L(0) e^{-\alpha t}$$

(Example) $R = 40\,\Omega$, $L = 20\,H$, $C = \frac{1}{20}F$, $i_L(0) = 5$, $i_L'(0) = 0$

then $\frac{R}{2L} = \frac{1}{LC} = \alpha = 1$

then $i_L(t) = 5\times 1\, t e^{-t} + 5 e^{-t} = 5(t+1)e^{-t}$



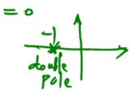
$20 i_L + 10 \frac{di_L}{dt} + \frac{1}{1/10}\int i_L(\tau)\, d\tau = 0$

$20 \frac{di_L}{dt} + 10 \frac{d^2 i_L(t)}{dt^2} + 10 i_L(t) = 0$

for $i_L(t) = e^{st}$, characteristic eq.

$20 s e^{st} + 10 s^2 e^{st} + 10 e^{st} = 0$

$\Rightarrow s^2 + 2s + 1 = (s+1)^2 = 0$



$$i_L(t) = A_1 t e^{-t} + A_2 e^{-t}$$

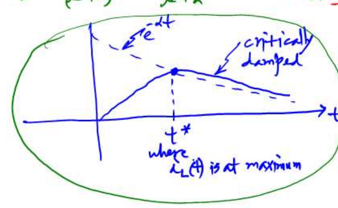
$\boxed{i_L(0) = A_2 = 5}$

$i_L'(0) = A_1 \cdot e^{-t} + A_1 t(-1) e^{-t} - A_2 e^{-t} \Big|_{t=0}$

$= A_1 - A_2 = 0 \quad \boxed{A_1 = A_2}$

$i_L(t) = 5 t e^{-t} + 5 e^{-t} = 5(t+1)e^{-t}$ ans

$$I_L(s) = \frac{K_1}{(s+\alpha)^2} + \frac{K_2}{s+\alpha} \xrightarrow{\mathcal{L}^{-1}} i_L(t) = K_1 t e^{-\alpha t} + K_2 e^{-\alpha t}$$



$i_L(0) = K_2$

Also i_L^{max} occurs when $\frac{di_L(t)}{dt} = 0$

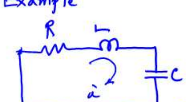
$\Rightarrow \boxed{K_1 e^{-\alpha t} + K_1 t(-\alpha) e^{-\alpha t} + K_2(-\alpha) e^{-\alpha t} = 0}$

$\Rightarrow K_1 - K_1\alpha t - K_2\alpha = 0$

thus $t^* = \frac{K_1 - K_2\alpha}{K_1\alpha} = \frac{1}{\alpha} - \frac{K_2}{K_1}$

where $K_1 = i_L'(0) + i_L(0)\alpha$

Example



$R = 40\,\Omega$

$L = 4\,H$

$C = \frac{1}{4}F$

$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\, dt = 0$

$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i = 0$

$(Ls^2 + Rs + \frac{1}{C}) I(s) = F(s)$

$I(s) = \frac{F(s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$

$i(0) = 1$

$v(0) = 0$

$40 + 4\frac{di(0)}{dt} + 0 = 0$

$i'(0) = -10$

1) Calculate the characteristic roots of the circuit. $s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 10s + 1 = 0$

2) Is it overdamped, underdamped or critically damped?

$(s+5)^2 - 24 = 0$

$s = -5 \pm \sqrt{24}$

$= -5 \pm 2\sqrt{6}$

$= -5 \pm 2(2.45)$

$= -0.5, -9.5$

$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

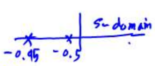
α ω_0^2

$\frac{R}{2L} = \frac{40}{2\times 4} = 5 = \alpha$

$\frac{1}{LC} = \frac{1}{4\times\frac{1}{4}} = 1 = \omega_0^2$

$= -5 \pm \sqrt{5^2 - 1}$

$\alpha^2 - \omega^2 = 25 - 1 = 24 > 0$ (overdamped)



$$i(t) = A_1 e^{-0.5t} + A_2 e^{-9.5t}$$

$i(0) = A_1 + A_2$

$i'(0) = -0.5A_1 - 9.5A_2 = -10$

$A_1 + A_2 = 1 \quad (1)$

$-0.5A_1 - 9.5A_2 = -10 \quad (2)$

$\times 2 \rightarrow$

$-A_1 - 19A_2 = -20 \quad (3)$

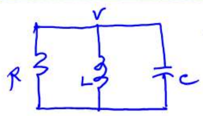
$(1)+(3) \Rightarrow 0 - 18A_2 = -19 \qquad A_2 = \frac{19}{18}$

$A_1 = -\frac{1}{18}$

$-A_1 - 19A_2 = +\frac{1}{18} - 19\left(\frac{19}{18}\right) = \frac{1-361}{18} = -20$ ✓

$$i(t) = -\frac{1}{18} e^{-0.5t} + \frac{19}{18} e^{-9.5t}$$ ans

Parallel RLC circuit



In series RLC circuit the loop current is common to all elements. (same)

In parallel RLC circuit, the node voltage is common. (same)

$$KCL \Rightarrow \frac{v}{R} + \frac{1}{L}\int v(\tau)\,d\tau + C\frac{dv}{dt} = 0$$

$$\text{Taking } \frac{d}{dt} \rightarrow \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0$$

$$\text{dividing by } C \rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

$$\text{For } v = e^{st}, \quad \left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right]e^{st} = 0$$

series ckt

$$\text{Thus } \boxed{s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0} \leftrightarrow \boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

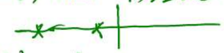
Let $\alpha = \frac{1}{2RC}$, $\omega_0 = \frac{1}{\sqrt{LC}}$

Then $s^2 + 2\alpha s + \omega_0^2 = 0$

$$(s+\alpha)^2 - (\alpha^2 - \omega_0^2) = 0$$

$$\left(s+\alpha+\sqrt{\alpha^2-\omega_0^2}\right)\left(s+\alpha-\sqrt{\alpha^2-\omega_0^2}\right) = 0 \quad \boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2-\omega_0^2}}$$

1) If $\alpha > \omega_0$, then s_1, s_2 are both negative, real.



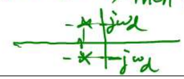
Overdamped

2) If $\alpha = \omega_0$, then $s_1, s_2 = -\alpha$ (double root)

critically damped

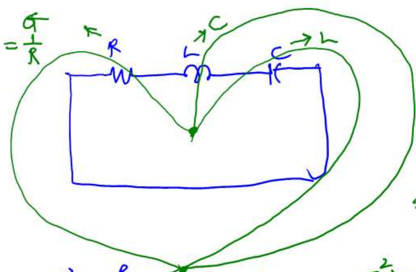
3) If $\alpha < \omega_0$, then $s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$

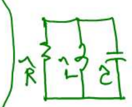
$= -\alpha \pm j\omega_d$



underdamped

Dual circuit





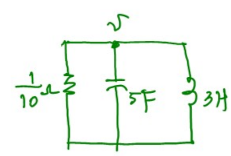
Dual of series RLC

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad s^2 + \frac{1}{\hat{R}\hat{C}}s + \frac{1}{\hat{L}\hat{C}} = 0$$

$$\frac{R}{L} = \frac{1}{\hat{R}\hat{C}} \qquad \frac{1}{LC} = \frac{1}{\hat{L}\hat{C}}$$

$$\Rightarrow \boxed{\hat{C} = L, \ \hat{R} = \frac{1}{R}, \ \hat{C} = L}$$

Example



$$10i + 5\frac{di}{dt} + \frac{1}{3}\int i(\tau)\,d\tau = 0$$

$$\frac{v}{(1/10)} + 5\frac{dv}{dt} + \frac{1}{3}\int v(\tau)\,d\tau = 0$$

$$\boxed{5\frac{d^2i}{dt^2} + 10\frac{di}{dt} + \frac{1}{3}i = 0}$$

$$10v + 5\frac{dv}{dt} + \frac{1}{3}\int v(\tau)\,d\tau = 0$$

$$\boxed{5\frac{d^2v}{dt^2} + 10\frac{dv}{dt} + \frac{1}{3}v(t) = 0}$$

$$5s^2 + 10s + \frac{1}{3} = 0 \quad \longleftrightarrow \quad 5s^2 + 10s + \frac{1}{3} = 0$$

same!

TABLE 8.1

Dual pairs.

Resistance R	Conductance G
Inductance L	Capacitance C
Voltage v	Current i
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Two circuits that are described by equations of the same form, but in which the variables are interchanged, are said to be dual to each other.

Two circuits are said to be **duals** of one another if they are described by the same characterizing equations with dual quantities interchanged.